# Autocorrelation function of level velocities for ray-splitting billiards

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We study experimentally and theoretically the autocorrelation function of level velocities $c(x)$ and the generalized conductance $C(0)$ for classically chaotic ray-splitting systems. Experimentally, a Sinai ray-splitting billiard was simulated by a thin microwave rectangular cavity with a quarter-circle Teflon insert. For the theoretical estimates of the autocorrelator $c(x)$ and the conductance $C(0)$ we made parameter-dependent quantum calculations of eigenenergies of an annular ray-splitting billiard. Our experimental and numerical results are compared to theoretical predictions of systems based on the Gaussian orthogonal ensemble in random matrix theory.

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Many statistical properties of quantum spectra of systems which are classically chaotic can be interpreted in terms of random matrix theory (RMT). RMT, initiated by Wigner and mainly developed by Dyson [1] and Mehta [2], considers ensembles of Hamiltonians and predicts universal statistical behavior of the corresponding quantal spectra. Recently, studies of universal properties of velocity autocorrelation functions of Hamiltonian systems that depend on a parameter $X$, such as a magnetic field [3–9] or the shape of a billiard [10], have attracted theoretical attention [3–10]. The velocity autocorrelation function has also been studied for random matrix dynamics [11–14].

The autocorrelator of the level velocities, originally introduced by Yang and Burgdörfer [15], is given by

$$c(x)=\frac{C(X)}{C(0)}, \tag{1}$$

where

$$C(X)=\frac{1}{\Delta^2}\left[\left\langle \frac{dE_i(\bar{X}+X)}{d\bar{X}}\,\frac{dE_i(\bar{X})}{d\bar{X}}\right\rangle-\left\langle\frac{dE_i(\bar{X})}{d\bar{X}}\right\rangle^2\right]$$

and

$$x=X\sqrt{C(0)}.$$

Szafer and Altshuler [3] and Simons and Altshuler [4] have shown that $c(x)$ is a universal function for all Hamiltonians which are members of the Gaussian orthogonal ensemble (GOE) or the Gaussian unitary ensemble and which depend on an external parameter $X$. $E_i$ and $\Delta$ denote the $i$th eigenenergy of the Hamiltonian system and the mean level spacing, respectively. The statistical averaging denoted by $\langle\ \rangle$ can be carried out over the energy levels $E_i$ or/and over a representative range of $\bar{X}$. The scaling parameter $C(0)=C(X)|_{X=0}$ is interpreted as a generalized conductance [4] and depends on the statistical properties and geometry of the system [9,10].

In this paper we study experimentally and numerically the autocorrelation functions of level velocities $c(x)$ in a new class of quantum systems—ray-splitting (RS) billiards [16–21]. Ray splitting occurs in many fields of physics, whenever a wave length is small in comparison with the range over which a potential changes. Ideal model systems for the investigation of ray-splitting phenomena are ray-splitting billiards [18,20,21] and microwave cavities with dielectric inserts [22,23]. An important new aspect of the recent investigation of ray-splitting is that the underlying classical mechanics in ray-splitting billiards is a non-Newtonian and nondeterministic mechanics [18,20]. The signature of non-Newtonian orbits has been found in the spectra of quantum ray-splitting billiards [18,20,21,24] and in the spectra of dielectric-loaded microwave cavities [22,23]. In this paper we investigate the autocorrelator of the level velocity of the annular ray-splitting billiard [21] numerically and the Teflon-loaded Sinai microwave cavity experimentally.

Recent studies of non-RS systems [7,10,25,26] show deviations of $c(x)$ from the GOE expectations. The aim of our paper is to extend the investigation of $c(x)$ to ray splitting systems in order to check whether the deviations persist.

The Sinai microwave cavity consists of a thin microwave cavity of dimensions $h=0.8$ cm (height), $w=20$ cm (width) with a quarter-circle Teflon insert of radius $r=7$ cm (see Fig. 1). It is well known that the electrodynamics of a thin microwave cavity can be described by the Helmholtz equation [27], which is equivalent to the Schrödinger equation in a two-dimensional quantum billiard [28]. In [29,30] it was shown that for frequencies $\nu$ less than a cutoff frequency $\nu_c$ the frequency spectrum of a dielectric-loaded microwave cavity is equivalent to the quantum spectrum of a corresponding two-dimensional ray-splitting system. The cutoff frequency is given by $\nu_c=c/(2hn)$, where $c$ is the speed of light and $n$ is the index of refraction of the dielectric insert.

 

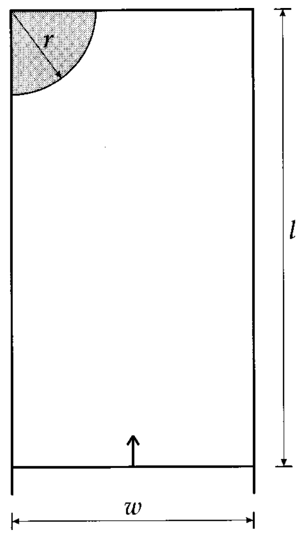


FIG. 1. Sketch of the Sinai microwave cavity. The width of the cavity is $w=20$ cm and the length $l$ varies between $l_i=36.9$ cm and $l_f=39.2$ cm. A quarter-circle Teflon insert ($r=7$ cm) with the same height as the cavity (0.8 cm) is inserted in the microwave cavity.

For a Teflon-loaded microwave cavity the index of refraction is $n\simeq1.44$ [31] and the cutoff frequency is $\nu_c\simeq13$ GHz.

To generate level dynamics one has to choose an appropriate system parameter that can be controlled easily. We have changed the length $l$ of the Sinai microwave cavity in the range from $l_i=36.9$ cm to $l_f=39.2$ cm in steps of 0.05 cm and measured resonance frequencies $\nu_j$, $j=1,\ldots,364$ as a function of the parameter $l$ for the frequency range from 0.5 GHz to 12 GHz. The cavity's spectra were measured using a frequency step of 0.4 MHz. We checked that this step was small enough to resolve all of the details of the spectra. The corresponding wave numbers are $k_j(l)=2\pi\nu_j(l)/c$ and the energies are $E_j(l)=k_j^2(l)$. A typical set of energy levels of the Sinai microwave cavity as a function of $l$ is shown in Fig. 2, from which it is evident that the level dynamics is irregular and shows level repulsion.

The annular ray-splitting billiard (see Fig. 3) is derived

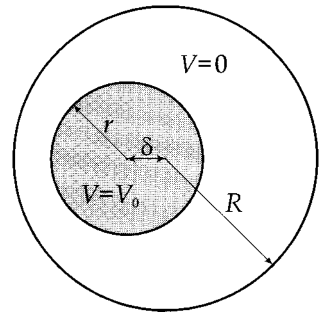


FIG. 3. Annular RS billiard. The radii of the outer and the inner circles are $R=1$ and $r=0.5$, respectively. The distance between the centers of the circles is denoted by $\delta$.

from the annular billiard studied by Bohigas *et al.* [32]. It consists of a disc of radius $a$ held at constant potential $V=V_0$ completely embedded in a circular domain of radius $R=1$ held at constant potential $V=0$. The distance of the centers of the circles is denoted by $\delta$, with $a>\delta$. We considered a particle of mass $m=1/2$ inside the potential of the annular ray-splitting billiard shown in Fig. 3 with Dirichlet boundary conditions imposed on the outer circle. The quantum dynamics of this system was solved numerically using the high accuracy method described in detail in [21]. We chose $\hbar=1$, $a=0.5$, and $\eta=V_0/E=1/2$ and calculated the first 75 scaled states of negative parity for different values of $\delta$. This way we generated the level dynamics as a function of the parameter $\delta$. The displacement parameter $\delta$ was varied from $\delta_i=0.16$ to $\delta_f=0.399$ in steps of 0.001. A set of levels of the annular ray-splitting billiard as a function of the parameter $\delta$ is shown in Fig. 4. As in the case of the Sinai microwave cavity the level dynamics is irregular and shows level repulsion.

To test the autocorrelator of the level velocities we chose $l-l_i$ and $\delta-\delta_i$ as the external parameter $X$ for the Sinai RS microwave cavity and the annular ray-splitting billiard, respectively. In order to calculate $c(x)$ for the Sinai microwave cavity, we used the levels $\nu_j$, $j=101,\ldots,364$, because the first 100 levels show a nonuniversal behavior. The lowest of the discarded levels display a very weak dependence on the cavity length $l$. The remaining spectrum was divided into

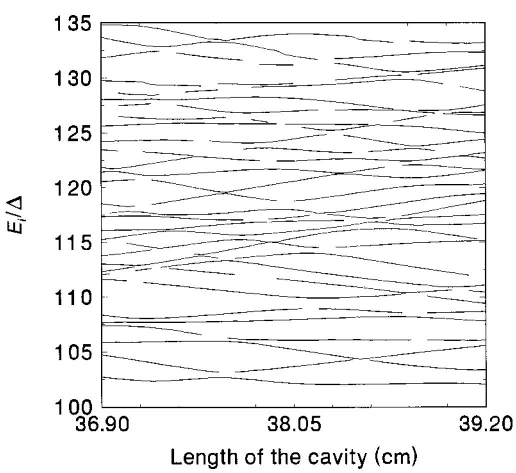


FIG. 2. Range of the spectrum of the Sinai RS billiard. The spectrum was unfolded using the Weyl formula for the microwave cavity including the discontinuity in the dielectric constant [17,23].

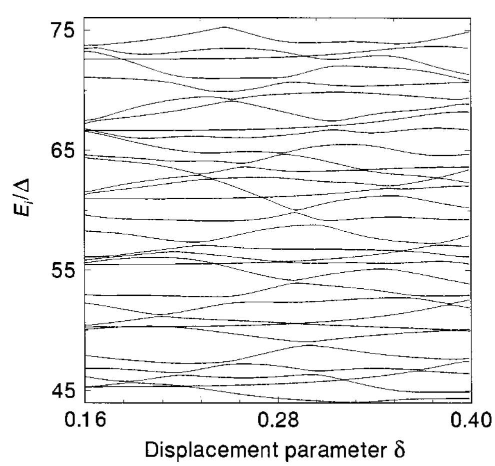


FIG. 4. Spectrum of the annular RS billiard for the eigenenergies $E_i, i=45-75$. The spectrum was unfolded using the Weyl formula for the annular RS billiard [21].

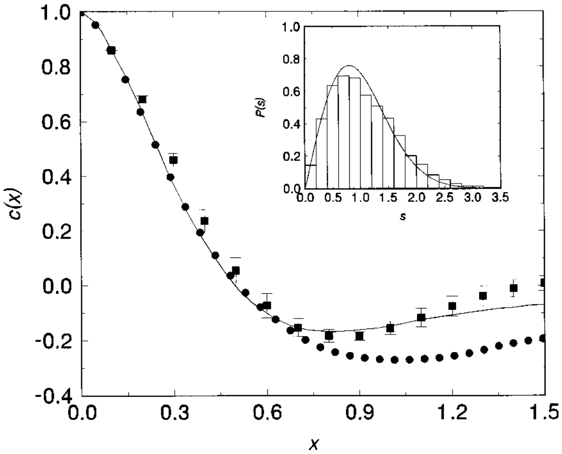


FIG. 5. Autocorrelation function of the level velocities $c(x)$ of the Sinai microwave cavity (squares) and the annular RS billiard (circles) in comparison with the theoretical prediction for GOE [10]. The inset shows the nearest-neighbor distribution for the Sinai RS cavity compared to GOE prediction.

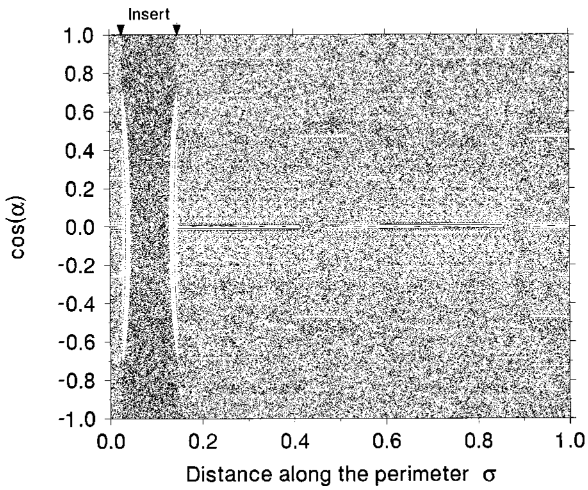


FIG. 6. Poincaré section of the electromagnetic ray dynamics (zero wavelength limit) for the Sinai RS cavity. Size of the cavity: width = 20 cm, length = 37.4 cm, radius of the Teflon insert $r = 7$ cm. Coordinates: $\sigma$, distance along the perimeter normalized to the length of the perimeter; $\cos(\alpha)$, the cosine of the bounce angle $\alpha$ (see Fig. 2 in [19]). The position of the Teflon insert along the perimeter of the Sinai RS cavity is marked with arrows.

three pieces (each piece containing 88 levels) because it changes as a function of energy. Within each piece of the spectrum the autocorrelator $c_n(x)$, $n=1,2,3$ was independently calculated and the mean value $c(x)=\frac{1}{3}\Sigma_{n=1}^{3}c_n(x)$ and its error were estimated. In the calculation of $c(x)$ for the annular ray-splitting billiard we used 31 eigenenergies $E_i$, $i=45,\ldots,75$ and omitted the first 44 levels for the same reasons as given above. From a statistical point of view, the size of the numerical data set is large enough, because we use 7409 velocities $[dE(\overline{X})]/(d\overline{X})$ to calculate the autocorrelator $c(x)$. Calculations performed for the Sinai RS cavity (annular RS billiard) showed that the generalized conductance $C(0)$ is a nonmonotonic, weak function of the cavity length $l$ (displacement parameter $\delta$). Therefore, in the calculation of $c(x)$ the statistical averaging was carried out over the full range of the parameter $\overline{X}$, e.g., for the Sinai cavity from $\overline{X}=l_i$ to $\overline{X}=l_f$. Figure 5 shows the autocorrelator of the level velocities $c(x)$ for the Sinai microwave cavity (squares) and for the annular RS billiard (circles) in comparison to the result predicted by RMT for GOE (full line) [10]. For small values of $x$, the experimental result and the numerical result are in good agreement with the RMT predictions. For larger values of $x$, both the experimental and the numerical result show deviations from the predictions of RMT, wherein the deviation of the result for the annular ray-splitting billiard is larger. The deviation of the result for the annular ray-splitting billiard may be explained by the presence of regular regions in the phase-space [21]. In [21] the existence of regular regions in the phase space of the annular ray-splitting billiard was also cited as a reason for the deviation of the nearest-neighbor spacing distribution from a Wigner statistic. Nevertheless, the result obtained for the two-dimensional conformal billiard [10] shows the same downward deviations from the result predicted by RMT, although it was shown that the two-dimensional conformal billiard is fully chaotic [33]. The diamagnetic Kepler problem [7] shows similar deviations in the downward direction, and Simons *et al.* [7] explained this by the presence of quasiregular features of the spectrum for large values of $x$. Also for the Sinai quartz block [25] the autocorrelator $c(x)$ shows deviations in the downward direction around $x=0.8$. The discrepancy was explained by the non-GOE behavior of the experimental data. Recently, measurements of the autocorrelator $c(x)$ have been reported for a conventional microwave Sinai cavity (rectangular cavity with a metallic insert in the shape of a quarter circle) and for the rectangular cavity with scatterers [26]. Although the overall agreement between the experiment with the Sinai cavity and theory is good, $c(x)$ shows deviations in the upward direction for $x=0.3-0.9$. For the local parameter variation (shift of one small scatterer inside the rectangular cavity containing additionally 19 randomly distributed small scatterers), the deviation of the autocorrelator $c(x)$ from RMT theory is much stronger than in the case of the global parameter variation (e.g., shift of a billiard wall) considered in our paper (Fig. 4 in [26]). For small and large values of $x$ the autocorrelator of the level velocities for the Sinai RS microwave cavity is in good agreement with the predictions of RMT (see Fig. 5), although for large values of $x$ there are small upward deviations. In contrast to the nearest-neighbor distribution for the annular ray-splitting billiard, the nearest-neighbor distribution for the Sinai microwave cavity is close to Wigner's surmise as shown by the inset in Fig. 5. The nearest-neighbor distribution for the RS Sinai billiard strongly suggests that the underlying ray dynamics of this system is chaotic. The Poincaré section of the electromagnetic ray dynamics (zero wavelength limit) for the Sinai RS cavity is shown in Fig. 6. This Poincaré section is generated by taking ray-splitting effects into account. When the ray strikes one of the outer edges of the cavity it is specularly reflected. When it strikes an interface between regions with different indices of refraction it has a probability $\mathcal{T}$ of being transmitted (transmitivity) and a probability $\mathcal{R}$ of being reflected (reflectivity) [35]. We use a Monte Carlo approach in order to avoid the ''daughter'' ray generation at each encounter with the RS interface [16]. When the ray hits the interface, whether it is transmitted or reflected is chosen randomly, according to probabilities $\mathcal{T}$ and $\mathcal{R}$. In this way only one ray is traced. The direction of the transmitted ray is given by Snell's law. Two vertical stripes of regular motion are present in the Poincaré section. They correspond to the trajectories that skip along the curved inside of the Teflon insert. Quantum mechanically they correspond to whispering gallery modes [18]. Outside of the Teflon insert we see a sea of chaos with some small regular structures. The central structures are due to marginally stable bouncing ball orbits. The Poincaré section shows that the ray-splitting effects resulting in the appearance of reflected and transmitted rays cause the ray dynamics of this system to become almost completely chaotic.

On the basis of the discussion presented in this paper, we conclude that the origin of the discrepancies between the predictions of RMT and calculations and measurements of the autocorrelator $c(x)$ for the classically chaotic quantum systems investigated in [7,10,25,26] and in this paper is not yet fully understood and is possibly connected to some degree of nonuniversality in the spectra of these systems.

Our experimental and numerical results also allow us to check the scaling properties of the parameter $C(0)$ as a function of the number of energy levels $N$. Recently, Bruus *et al.* [10] evaluated the scaling parameter $C(0)$ for the conformal billiard (a GOE system). Changing the shape of the conformal billiard, they found a scaling with the energy $E$ according to $C(0)\sim E^{3/2}$. Using the leading term in the Weyl formula [28] $N\simeq AE/4\pi$, where $A$ is the area of the billiard, we obtain the relation $C(0)\sim N^{3/2}$. The scaling properties of $C(0)$ for the RS billiards are analyzed based on the leading term in the Weyl formula, which according to [17,23] does not depend on the RS phenomena. Figure 7 shows the parameter $C(0)$ as a function of $N$ for the Sinai microwave cavity. The parameter $C(0)$ is averaged over 21 neighboring levels. We performed a least squares fit for $C(0)$ according to

$$C(0)=aN^b, \tag{2}$$

and obtained $a=26.8\ \mathrm{m}^{-2}\pm 6.3\ \mathrm{m}^{-2}$ and $b=1.47\pm 0.10$. Although strong fluctuations are present in $C(0)$, possibly caused by the bouncing ball orbits of the Sinai billiard [34], the exponent $b$ is very close to the value of 3/2 predicted for non-RS billiards [10].

The inset in Fig 7 shows the variation of $C(0)$ as a function of $N$ for the annular RS billiard. In this case, too, the parameter $C(0)$ is averaged over 21 neighboring states. A least squares fit for the case of the annular ray-splitting billiard yielded $a=1.03\pm 0.27$ and $b=1.38\pm 0.15$. The exponent $b$ coincides with the value 3/2 within the error limits.

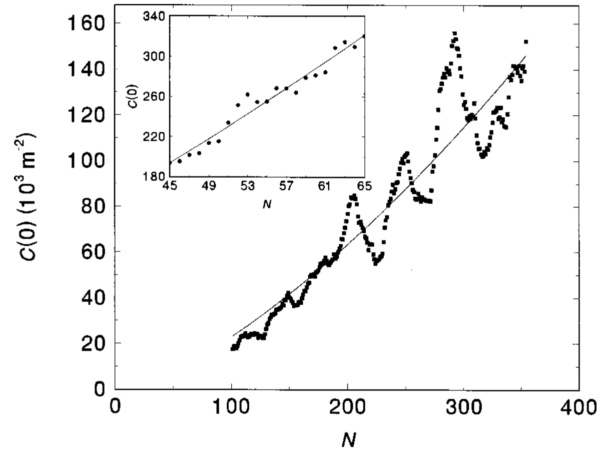


FIG. 7. Scaling parameter $C(0)$ for the Sinai microwave cavity as a function of the energy level number $N$. The full line is given by the least squares fit: $C(0)=aN^b$, with $a=26.8\ \mathrm{m}^{-2}\pm 6.3\ \mathrm{m}^{-2}$ and $b=1.47\pm 0.10$. The inset shows the scaling parameter $C(0)$ for the annular RS billiard. The full line is given by the least squares fit: $C(0)=aN^b$, with $a=1.03\pm 0.27$ and $b=1.38\pm 0.15$.

In summary, we measured and calculated the autocorrelator of the level velocities $c(x)$ and the scaling parameter $C(0)$ for the Sinai microwave cavity and the annular ray-splitting billiards. For the Sinai RS billiard we found good agreement of the estimated $c(x)$ with the GOE predictions. For $x>0.6$ the correlator $c(x)$ calculated for the annular RS billiard is below the GOE results. Thus, the main result of our paper is that the deviations in $c(x)$ persist for RS systems. This behavior may be linked to nonuniversal, quasiregular eigenenergies still present in the set of energies used for calculating the autocorrelator $c(x)$. In both cases, however, we found that the scaling of the parameter $C(0)$ is close to the prediction $C(0)\sim N^{3/2}$ obtained for non-RS billiards [10]. Our results for the autocorrelator $c(x)$ and the scaling parameter $C(0)$ suggest that as far as the properties of parametrically dependent eigenenergies of classically chaotic quantum systems are concerned, there is no essential difference between RS and non-RS systems.

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[1] F.J. Dyson, J. Math. Phys. **3**, 140 (1962).
[2] M.L. Mehta, *Random Matrices* (Academic Press, New York, 1991).
[3] A. Szafer and B.L. Altshuler, Phys. Rev. Lett. **70**, 587 (1993).
[4] B.D. Simons and B.L. Altshuler, Phys. Rev. Lett. **70**, 4063 (1993).
[5] B.D. Simons and B.L. Alshuler, Phys. Rev. B **48**, 5422 (1993).
[6] M. Faas, B.D. Simons, X. Zotos, and B.L. Altshuler, Phys. Rev. B **48**, 5439 (1993).
[7] B.D. Simons, A. Hashimoto, M. Courtney, D. Klepner, and B.L. Alshuler, Phys. Rev. Lett. **71**, 2899 (1993).
[8] M.V. Berry and J.P. Keating, J. Phys. A **27**, 6167 (1994).
[9] H. Bruus, C.H. Lewenkopf, and E.R. Mucciolo, Phys. Rev. B **53**, 9968 (1996).
[10] H. Bruus, C.H. Lewenkopf, and E.R. Mucciolo, Phys. Scr. **T69**, 13 (1997).

[11] D. Delande and J. Zakrzewski, J. Phys. Soc. Jpn. **63**, 101 (1994).

[12] J. Zakrzewski, Ż. Phys. B: Condens. Matter **98**, 273 (1995).

[13] I. Guarneri, K. Życzkowski, J. Zakrzewski, L. Molinari, and G. Casati, Phys. Rev. E **52**, 2220 (1995).

[14] P. Kunstman, K. Życzkowski, and J. Zakrzewski, Phys. Rev. E **55**, 2446 (1997).

[15] X. Yang and J. Burgdörfer, Phys. Rev. A **46**, 2295 (1992).

[16] L. Couchman, E. Ott, and T.M. Antonsen, Jr., Phys. Rev. A **46**, 6193 (1992).

[17] R.E. Prange, E. Ott, T.M. Antonsen, Jr., B. Georgeot, and R. Blümel, Phys. Rev. E **53**, 207 (1996).

[18] R. Blümel, T.M. Antonsen, B. Georgeot, E. Ott, and R.E. Prange, Phys. Rev. Lett. **76**, 2476 (1996); Phys. Rev. E **53**, 3284 (1996).

[19] R.N. Oerter, E. Ott, T.M. Antonsen, Jr., and P. So, Phys. Lett. A **216**, 59 (1996).

[20] A. Kohler, G.H.M. Killesreiter, and R. Blümel, Phys. Rev. E **56**, 2691 (1997).

[21] A. Kohler and R. Blümel, Phys. Lett. A **238**, 271 (1998).

[22] L. Sirko, P.M. Koch, and R. Blümel, Phys. Rev. Lett. **78**, 2940 (1997).

[23] Sz. Bauch, A. Błędowski, L. Sirko, P.M. Koch, and R. Blümel, Phys. Rev. E **57**, 304 (1998).

[24] A. Kohler and R. Blümel, Phys. Lett. A **247**, 87 (1998).

[25] K. Schaadt, M.Sc. thesis, University of Copenhagen, 1997, p. 50; T. Bertelsen, C. Ellegaard, T. Guhr, K. Lindemann, M. Oxborrow, and K. Schaadt (unpublished).

[26] M. Barth, U. Kuhl, and H.-J. Stöckmann, Phys. Rev. Lett. **82**, 2026 (1999).

[27] J.D. Jackson, *Classical Electrodynamics* (John Wiley, New York, 1975).

[28] M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990).

[29] H.-J. Stöckmann and J. Stein, Phys. Rev. Lett. **64**, 2215 (1990).

[30] S. Sridhar, Phys. Rev. Lett. **67**, 785 (1991).

[31] K.H. Breeden and A.P. Sheppard, Microwave J. **10**, 59 (1967); Radio Sci. **3**, 205 (1968).

[32] O. Bohigas, D. Boosé, R. Egydio de Carvalho, and V. Marvulle, Nucl. Phys. A **560**, 197 (1993).

[33] H. Bruus and A.D. Stone, Phys. Rev. B **50**, 18 275 (1994); Physica B **189**, 43 (1993).

[34] M. Kollmann, J. Stein, U. Stoffregen, H.-J. Stöckmann, and B. Eckhardt, Phys. Rev. E **49**, R1 (1994).

[35] E. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Oxford, 1964).